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ON THE EULER CHARACTERISTIC OF SPHERICAL POLYHEDRA AND THE EULER RELATION

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Let E^{n+1}, for some integer $n \geqslant 0$, be the $(n+1)$-dimensional Euclidean space, and denote by S^n the standard n-sphere in E^{n+1}, $S^n := \{x \in E^{n+1} : \|x\| = 1\}$. It is convenient to introduce the (-1)-dimensional sphere $S^{-1} := \varnothing$, where $\varnothing$ denotes the empty set. By an i-dimensional subsphere T of S^n, $i = 0, \ldots, n$, we understand the intersection of S^n with some $(i+1)$-dimensional subspace of E^{n+1}. The affine hull of T always contains, with this definition, the origin of E^{n+1}. $\varnothing$ is the unique (-1)-dimensional subsphere of S^n. By the spherical hull, $\operatorname{sph} X$, of a set $X \subset S^n$, we understand the intersection of all subspheres of S^n containing X. Further we set $\dim X := \dim \operatorname{sph} X$. The interior, the boundary and the complement of an arbitrary set $X \subset S^n$, with respect to S^n, shall be denoted by $\operatorname{int} X$, $\operatorname{bd} X$ and $\operatorname{cpl} X$. Finally we define the relative interior $\operatorname{rel\,int} X$ to be the interior of $X \subset S^n$ with respect to the usual topology of $\operatorname{sph} X \subset S^n$. For $n \geqslant 1$ each $(n-1)$-dimensional subsphere of S^n defines two closed hemispheres of S^n, whose common boundary it is. The two hemispheres of the sphere S^0 are defined to be the two one-pointed subsets of S^0. A subset $P \subset S^n$ is called a closed (spherical) polytope, if it is the intersection of finitely many closed hemispheres, and, if, in addition, it does not contain a subsphere of S^n. $Q \subset S^n$ is called an i-dimensional, relatively open polytope, $i \geqslant 1$, or shortly an i-open polytope, if there exists a closed polytope $P \subset S^n$ such that $\dim P = i$ and $Q = \operatorname{rel\,int} P$. $X \subset S^n$ is called a closed polyhedron, if it is a finite union of closed polytopes $P_1, \ldots, P_r$. The empty set $\varnothing$ is the only (-1)-dimensional closed polyhedron of S^n. We denote by $\mathfrak{X}$ the set of all closed polyhedra of S^n. $Y \subset S^n$ is called an i-open polyhedron, for some $i \geqslant 1$, if there are finitely many i-open polytopes $Q_1, \ldots, Q_r$ in S^n such that $Y = Q_1 \cup \ldots \cup Q_r$, and $\dim Y = i$. By $\mathfrak{Y}_i$ we denote the set of all i-open polyhedra. Clearly $\varnothing \in \mathfrak{X}$, $\varnothing \notin \mathfrak{Y}_i$, for all $i \geqslant 1$, and each i-dimensional subsphere of S^n, $i \geqslant 1$, belongs to $\mathfrak{X}$ and to $\mathfrak{Y}_i$. For each i-dimensional subsphere T of S^n, set $\mathfrak{Y}_i(T) := \{T \in \mathfrak{Y}_i : Y \subset T\}$. A map $\varepsilon : \mathfrak{X} \cup \mathfrak{Y}_1 \cup \ldots \cup \mathfrak{Y}_n \to \{0, 1\}$ is defined by $\varepsilon X := 0$, for all $X \in \mathfrak{X}$, and $\varepsilon Y := 1$, for all $Y \in \mathfrak{Y}_1 \cup \ldots \cup \mathfrak{Y}_n$, $Y \notin \mathfrak{X}$.

DEFINITION 1. *Let $\mathfrak{Z}$ be a ring of subsets of S^n, generated by some subset of $\mathfrak{X} \cup \mathfrak{Y}_1 \cup \ldots \cup \mathfrak{Y}_n$. An Euler characteristic on $\mathfrak{Z}$ is a map $\psi : \mathfrak{Z} \to \mathbb{Z}$ (the ring of* *integers) with the following properties*:

(1) *If* $\varnothing \in \mathfrak{Z}$, *then* $\psi\varnothing = 0$.

(2) $\psi X = 1$, *whenever* X *is a closed non-void polytope, or an i-open polytope* $(i \geqslant 1)$, *contained in* $\mathfrak{Z}$.

(3) *For all* X, Y *in* $\mathfrak{Z}$, $\psi(X \cup Y) + \psi(X \cap Y) = \psi X + \psi Y$.

It is well known that there exists a unique Euler characteristic χ_0 on $\mathfrak{X}$, and, for each i-dimensional subsphere T of S^n, a unique Euler characteristic χ_T on $\mathfrak{Y}_i(T)$ (see [**2**], [**3**]). For notational convenience we denote all these characteristics by the same letter χ. Thus a mapping $\chi : \mathfrak{X} \cup \mathfrak{Y}_1 \cup \ldots \cup \mathfrak{Y}_n \to \mathbb{Z}$ is defined, which satisfies (1) and (2), and which satisfies (3) for certain pairs of polyhedra. On the other hand we notice that there are rings $\mathfrak{Z}$ which admit no Euler characteristic, and others which admit more than one. For example there exists no Euler characteristic on the ring of sets generated by $\mathfrak{X} \cup \mathfrak{Y}_1 \cup \ldots \cup \mathfrak{Y}_n$, $n \geqslant 1$. To see this, consider a 1-dimensional subsphere $S \subset S^n$, a set $X \subset S$ with two elements, and the complement $Y := S \sim X$. (3) would not hold for X and Y. Sometimes it is more convenient to study the map $\omega : \mathfrak{X} \cup \mathfrak{Y}_1 \cup \ldots \cup \mathfrak{Y}_n \to \mathbb{Z}$ defined by $\omega(U) := (-1)^{\epsilon U \dim U}\chi(U)$, rather than χ itself. For $n \geqslant 1$, let $S \subset S^n$ be a subsphere of dimension $n-2$, and denote by $\mathfrak{S}$ the set of all $(n-1)$-dimensional subspheres of S^n containing S, together with the usual topology. $\mathfrak{S}$ is homeomorphic to the real projective line, and hence to S^1. Each choice of an orientation of $\mathfrak{S}$ and of a fixed element $S_0 \in \mathfrak{S}$ determines, by means of the "angular parameter", a continuous and periodic map $p : \mathbb{R} \to \mathfrak{S}$ with $p(t) = p(t+\pi)$, for each real number t, and with the fundamental interval $I := [0, \pi)$. For the rest of this article we assume that a fixed choice of the covering projection p has been made, for every $(n-2)$-dimensional subsphere $S \subset S^n$. The sphere $p(t) \in \mathfrak{S}$ will often be denoted by S_t. Given a map $f : \mathfrak{S} \to \mathbb{R}$ and an element $t \in I$, we define the right-hand limit $f^+(S_t)$ in the usual way. If there exists a real number x such that for each sequence of numbers t_n with $t_n \geqslant t$ and $t_n \to t$ $(n \to \infty)$ we have $fp(t_n) \to x$ $(n \to \infty)$, we set $f^+(S_t) := x$. We say that two subspheres S and T of S^n are in general position, if either $S \cap T = \varnothing$ or $\dim(S \cap T) = \dim S + \dim T - n$.

PROPOSITION 1. *Let* $X \subset S^n$, $n \geqslant 1$, *be a spherical polyhedron*,

$$X \in \mathfrak{X} \cup \mathfrak{Y}_1 \cup \ldots \cup \mathfrak{Y}_n,$$

and let $S \subset S^n$ *be an* $(n-2)$-*dimensional subsphere. With the notation introduced above*,

(i) $\omega X = \omega(X \cap S) + \sum_{t \in I} (\omega(X \cap S_t) - \omega^+(X \cap S_t))$.

As above $I := [0, \pi)$ is the fundamental interval of the periodic map $p : \mathbb{R} \to \mathfrak{S}$, where $\mathfrak{S}$ stands for the set of all $(n-1)$-spheres in S^n containing S. Before we proceed to prove Proposition 1, notice that the value $\omega(X \cap S_t) - \omega^+(X \cap S_t)$ vanishes for all but a single $t \in I$, whenever X is a closed polytope, or an i-open polytope, for some $i \geqslant 1$. Thus the sum to the right of the equality sign is in fact finite, for each polyhedron X. Proposition 1 is a spherical counterpart of a well known recursion formula for the Euler characteristic for Euclidean polyhedra (see [1]).

Proof of Proposition 1. We assume $X \in \mathfrak{Y}_i$, for some $i \geqslant 1$. The case $X \in \mathfrak{X}$ may be treated by an obvious modification of the argument. Set $R := \operatorname{sph} X$, and for each $Z \in \mathfrak{Y}_i(R)$,

$$\psi Z := (-1)^i\Big(\omega(Z \cap S) + \sum_{t \in I} (\omega(Z \cap S_t) - \omega^+(Z \cap S_t))\Big).$$

It suffices to show that ψ is an Euler characteristic on $\mathfrak{Y}_i(R)$. The requirements (1) and (3) of Definition 1 are satisfied by ψ. Now suppose that Z is an i-open polytope in R. Let us first assume $Z \cap S \neq \varnothing$. We distinguish three cases. If the spheres S and R are in general position we have $i \geqslant 2$, $\dim(Z \cap S) = i-2$, $\dim(Z \cap S_t) = i-1$, for each t in the interval $I := [0, \pi)$, hence $\psi Z = \chi(Z \cap S) = 1$. In the case $R \subset S$ we find $Z \cap S_t = Z \cap S = Z$, for every $t \in I$. This again implies $\psi Z = \chi(Z \cap S) = 1$. If none of the above cases hold we see that $R \not\subset S$, but $R \subset S_t$, for some number $t \in I$. Hence $Z \cap S_{t'} = Z \cap S$ for all $t' \in I$, $t' \neq t$, and

$$\psi Z = (-1)^i(\omega(Z \cap S) + \omega(Z \cap S_t) - \omega(Z \cap S)) = 1.$$

Assume now $Z \cap S = \varnothing$. We are confronted with two cases. If $R \subset S_t$, for some point $t \in I$, we have $Z \cap S_t = Z$ and $Z \cap S_{t'} = \varnothing$, for every $t' \in I$, $t' \neq t$. Clearly $\psi Z = 1$. If R and S are in general position, let $A \subset I$ be the set of all points $t \in I$ such that $Z \cap S_t \neq \varnothing$. A is an open interval in I, denote its left end-point by x. Clearly

$$\omega(Z \cap S_x) - \omega^+(Z \cap S_x) = -(-1)^{i-1},$$

whereas $\omega(Z \cap S_t) - \omega^+(Z \cap S_t) = 0$, for all $t \neq x$. This shows again $\psi Z = 1$, and ψ is indeed an Euler characteristic on $\mathfrak{Y}_i(R)$. To prove (3) for ψ, notice that $\chi(X) = 0$, for each odd dimensional sphere X, hence for each $X \in \mathfrak{Y}_{2k+1} \cap \mathfrak{X}$.

DEFINITION 2. *Let X be a spherical polyhedron, $X \in \mathfrak{X} \cup \mathfrak{Y}_1 \cup \ldots \cup \mathfrak{Y}_n$. By a δ-decomposition of X we understand a finite set $\mathfrak{D} \subset X \cup \mathfrak{Y}_1 \cup \ldots \cup \mathfrak{Y}_n$ such that $\bigcup \mathfrak{D} = X$, and, further, $U \cap V = \varnothing$ whenever U and V are two different members of $\mathfrak{D}$.*

If, for example, $\mathfrak{C}$ is a complex, in the usual sense of the word, whose members are closed spherical simplices, then the relative interiors of the elements of $\mathfrak{C}$ form a δ-decomposition of $\bigcup \mathfrak{C}$.

PROPOSITION 2. *For each spherical polyhedron $X \subset S^n$, $n \geqslant 1$,*

$$X \in \mathfrak{X} \cup \mathfrak{Y}_1 \cup \ldots \cup \mathfrak{Y}_n,$$

and for each δ-decomposition $\mathfrak{D}$ of X we have

(ii) $\omega X = \sum_{Y \in \mathfrak{D}} \omega Y.$

Proof. We proceed by induction on the dimension n of the sphere S^n containing X, the case $n = 0$ being trivial. For given $n \geqslant 1$, $X \in \mathfrak{X} \cup \mathfrak{Y}_1 \cup \ldots \cup \mathfrak{Y}_n$, and for a δ-decomposition $\mathfrak{D}$ of $X \subset S^n$, choose an $(n-2)$-sphere $S \subset S^n$. With the notation of the section preceding Proposition 1 we find, by Proposition 1 and the inductive assumption of our statement

$$\begin{aligned}
\omega X &= \omega(X \cap S) + \sum_{t \in I} (\omega(X \cap S_t) - \omega^+(X \cap S_t)) \\
&= \sum_{Y \in \mathfrak{D}} \omega(Y \cap S) + \sum_{t \in I} \sum_{Y \in \mathfrak{D}} (\omega(Y \cap S_t) - \omega^+(Y \cap S_t)) \\
&= \sum_{Y \in \mathfrak{D}} \Big(\omega(Y \cap S) + \sum_{t \in I} (\omega(Y \cap S_t) - \omega^+(Y \cap S_t))\Big) \\
&= \sum_{Y \in \mathfrak{D}} \omega Y.
\end{aligned}$$

As an application of the foregoing arguments let us derive some elementary relations involving the Euler characteristic.

PROPOSITION 3.

(iii) $\chi(S^n) = 1 + (-1)^n$

(iv) $\chi X = \chi(\operatorname{bd} X) + (-1)^n \chi(\operatorname{int} X)$ $\qquad X \subset S^n, \quad X \in \mathfrak{X}$

(v) $\chi(\operatorname{cpl} X) = 1 + (-1)^n - (-1)^n \chi X$ $\qquad X \subset S^n, \quad X \in \mathfrak{X}$

(vi) $\chi(\operatorname{cpl} Y) = 1 + (-1)^n - (-1)^n \chi Y$ $\qquad Y \subset S^n, \quad Y \in \mathfrak{Y}_n$.

Proof. (iii) We proceed by induction on n. The cases $n \leqslant 0$ are trivial. For $n \geqslant 1$ choose an arbitrary $(n-2)$-dimensional subsphere S of S^n, and apply Proposition 1 to the polyhedron $S^n \in \mathfrak{X}$. By the inductive hypothesis,

$$\chi S^n = \chi S = 1 + (-1)^{n-2} = 1 + (-1)^n.$$

(iv) $\{\operatorname{bd} X, \operatorname{int} X\}$ is a δ-decomposition of the polyhedron $X \in \mathfrak{X}$. By Proposition 2, $\omega X = \omega(\operatorname{bd} X) + \omega(\operatorname{int} X)$. Since $\{X, \operatorname{bd} X\} \subset \mathfrak{X}$ and $\operatorname{int} X \in \mathfrak{Y}_n$, our assertion follows at once from the definition of ω.

(v) $\{X, \operatorname{cpl} X\}$ is a δ-decomposition of the polyhedron $S^n \in \mathfrak{X}$. Our assertion follows immediately from Proposition 2 if we keep in mind that $\{X, S^n\} \subset \mathfrak{X}$ and $\operatorname{cpl} X \in \mathfrak{Y}_n$.

(vi) The proof of this relation is quite analogous to that of (v).

References

1. H. Hadwiger, " Eine Schnittrekursion für die Eulersche Charakteristik euklidischer Polyeder mit Anwendungen innerhalb der kombinatorischen Geometrie ", *El. Math.*, 23 (1968), 121–132.
2. V. Klee, " The Euler characteristic in combinatorial geometry ", *Amer. Math. Monthly*, 70 (1963), 119–127.
3. H. Lenz, " Mengenalgebra und Eulersche Charakteristik," *Abh. Math. Seminar Univ. Hamburg*, 34 (1970), 135–147.

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